NORM OF LOGARITHMIC PRIMARY OF VIRASORO ALGEBRA

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Abstract. We give an algebraic proof of the formula on the norm of logarithmic primary of Virasoro algebra, which was proposed by Al. Zamolodchikov. This formula appears in the recursion formula for the norm of Gaiotto state, which guarantees the AGT relation for the four-dimensional SU(2) pure gauge theory.

1. Introduction

This paper discusses algebraic or combinatorial calculation of certain elements in the Verma module of Virasoro algebra. Our main result is Theorem 1.2 stated in §1.6, where a mathematcal proof of the formula on the norm of logarithmic primary of Virasoro algebra is given. This formula was proposed in [47], and the proof [23] of the AGT relation for pure SU(2) gauge theory [19] depends on it. The result of this paper is the last piece of the proof of the AGT relation.

However, in order to state that, we need to introduce several notations and recall well-known facts on Virasoro algebra. Subsections §1.1 and §1.2 are devoted to these preliminaries. Such topics are often treated in the textbooks of conformal field theory, such as [15], [26] and [38]. The reader who is familiar with Virasoro algebra may skip to §1.3.

In §1.3 we introduce the norm of logarithmic primary, which is the main topic of this paper and appears directly in the statement of Theorem 1.2.

In §1.4 and §1.5 we mention to AGT conjectures/relations and its connection to our main theorem. These subsections are a detour, but it will be interesting for those working on AGT conjectures.

In the last subsection §1.6 of this introduction, we state our main theorem and the contents of the main part of this paper.

1.1. Virasoro algebra and singular vectors.

A singular vector in the Verma module of Virasoro algebra is a fundamental object in the two-dimensional conformal field theory and the representation theory of Virasoro algebra since its emergence in the classical paper [6].

Let us recall the definition of singular vectors, fixing notations on Virasoro algebra and its Verma module. The Virasoro algebra Vir is the Lie algebra

Date: October 17, 2010; revised June 11, 2011.

2010 *Mathematics Subject Classification.* 17B68, 05E05.

Key words and phrases. Virasoro algebra, singular vector, free field realization, Jack symmetric polynomials.

over $\mathbb{C}$ generated by L_n ($n \in \mathbb{Z}$) and C (central) with the relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}m(m^2-1)\delta_{m+n,0}, \quad [L_n, C] = 0. \tag{1.1}$$

Vir has the triangular decomposition $\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-$ with $\mathrm{Vir}_\pm := \oplus_{\pm n>0}\mathbb{C}L_n$ and $\mathrm{Vir}_0 := \mathbb{C}C \oplus \mathbb{C}L_0$.

Let c and h be complex numbers. Let $\mathbb{C}_{c,h}$ be the one-dimensional representation of the subalgebra $\mathrm{Vir}_{\geq 0} := \mathrm{Vir}_0 \oplus \mathrm{Vir}_+$, where Vir_+ acts trivially, L_0 acts as multiplication by h, and C acts as multiplication by c. Then the Verma module $M(c,h)$ is defined by

$$M(c,h) := \mathrm{Ind}_{\mathrm{Vir}_{\geq 0}}^{\mathrm{Vir}} \mathbb{C}_{c,h}.$$

Obeying the notation in physics literature, we denote by $|c,h\rangle$ a fixed basis of $\mathbb{C}_{c,h}$. Then one has $\mathbb{C}_{c,h} = \mathbb{C}\,|c,h\rangle$ and $M(c,h) = \mathrm{Vir}\,|c,h\rangle$.

$M(c,h)$ has an L_0-weight decomposition: $M(c,h) = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M(c,h)_n$ with

$$M(c,h)_n := \{v \in M(c,h) \mid L_0 v = (h+n)v\}.$$

A basis of $M(c,h)_n$ can be described by partitions. In this paper, a partition of positive integer n means a non-increasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\sum_{i=1}^k \lambda_i = n$. We also consider the empty sequence $\emptyset$ as a partition of the number 0. The symbol $\lambda \vdash n$ means that λ is a partition of n. We also define $|\lambda| := \sum_{i=1}^k \lambda_i$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n we use the symbol

$$L_{-\lambda} := L_{-\lambda_k} L_{-\lambda_{k-1}} \cdots L_{-\lambda_1}, \tag{1.2}$$

an element of the enveloping algebra $U(\mathrm{Vir}_-)$ of the subalgebra Vir_-. Using these notations, one finds that the set

$$\{L_{-\lambda}\,|c,h\rangle \mid \lambda \vdash n\},$$

is a basis of $M(c,h)_n$.

An element v of $M(c,h)_n$ is called a singular vector of level n if

$$L_k v = 0 \ \text{ for any } k \in \mathbb{Z}_{>0}. \tag{1.3}$$

The existence of singular vector restricts the values of the highest weights (c,h). To see this phenomena, it is necessary to recall the Kac determinant formula.

First, we introduce the (restricted) dual Verma module $M^*(c,h)$. This is a right Vir-module generated by $\langle c,h|$ with $\langle c,h|\,\mathrm{Vir}_+ = 0$, $\langle c,h|\,L_0 = h\,\langle c,h|$ and $\langle c,h|\,C = c\,\langle c,h|$. It has an L_0-weight decomposition $\bigoplus_{n\in\mathbb{Z}_{\geq 0}} M^*(c,h)_n$ with $M^*(c,h)_n := \{v \in M^*(c,h) \mid vL_0 = (h-n)v\}$. $M^*(c,h)_n$ has a basis $\{\langle h|\,L_\lambda \mid \lambda \vdash n\}$ with

$$L_\lambda := L_{\lambda_1} L_{\lambda_2} \cdots L_{\lambda_k}. \tag{1.4}$$

for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.

Next we introduce the contravariant form. It is a bilinear map on the modules

$$\cdot : M^*(c,h) \times M(c,h) \to \mathbb{C} \tag{1.5}$$

determined by

$$\langle h|\cdot|h\rangle = 1, \quad \langle h|\, u_1 u_2 \cdot |h\rangle = \langle h|\, u_1 \cdot u_2\, |h\rangle = \langle h| \cdot u_1 u_2\, |h\rangle \ \ (u_1, u_2 \in \mathrm{Vir}).$$

We usually omit the symbol $\cdot$ and write $\langle h|\, u\, |h\rangle := \langle h|\, u \cdot |h\rangle$ as in the physics literature. By counitng L_0-weights one can easily see that

$$M^*(c,h)_m \cdot M(c,h)_n = 0 \text{ if } m \neq n. \tag{1.6}$$

This bilinear form is contravariant in the following sense:

$$\langle h|\, L_\lambda L_{-\mu}\, |h\rangle = \langle h|\, L_\mu L_{-\lambda}\, |h\rangle \qquad \text{for any} \ \ \lambda,\ \mu. \tag{1.7}$$

It is usually called the contravariant form (or Shapovalov form) on the Verma module.

Recalling the basis (1.2) and the dual basis (1.4), we define

$$K_{\lambda,\mu}(c,h) := \langle c,h|\, L_\mu L_{-\lambda}\, |c,h\rangle \, .$$

Then the properties of the contravariant form are encoded in the (infinite size) matrix $(K_{\lambda,\mu})$, where λ and μ run over the set of all partitions. However, because of (1.6), we only need to consider the $p(n) \times p(n)$ matrix

$$K_n := (K_{\lambda,\mu})_{\lambda,\mu \vdash n} \tag{1.8}$$

for each $n \in \mathbb{Z}_{\geq 0}$. By (1.7) it is a symmetric matrix. Let us write down some examples:

$$\begin{aligned}
K_1 &= (K_{(1),(1)}) = (2h), \\
K_2 &= \begin{pmatrix} K_{(1^2),(1^2)} & K_{(1^2),(2)} \\ K_{(2),(1^2)} & K_{(2),(2)} \end{pmatrix} = \begin{pmatrix} 4h(1+2h) & 6h \\ 6h & 4h + c/2 \end{pmatrix}, \\
K_3 &= \begin{pmatrix} K_{(1^3),(1^3)} & K_{(1^3),(2,1)} & K_{(1^3),(3)} \\ K_{(2,1),(1^3)} & K_{(2,1),(2,1)} & K_{(2,1),(3)} \\ K_{(3),(1^3)} & K_{(3),(2,1)} & K_{(3),(3)} \end{pmatrix} \\
&= \begin{pmatrix} 24h(1+h)(1+2h) & 12h(1+3h) & 24h \\ 12h(1+3h) & 8h^2 + 8h + ch & 10h \\ 24h & 10h & 6h + 2c \end{pmatrix}.
\end{aligned} \tag{1.9}$$

The determinant $\det K_n$ is called the Kac determinant. As conjectured in [28] and shown in [13], [14], it has the factored form

$$\det K_n(c,h) = \prod_{\lambda \vdash n} 2^{\ell(\lambda)} z_\lambda \times \prod_{\substack{r,s \in \mathbb{Z}_{\geq 1} \\ rs \leq n}} \bigl(h - h_{r,s}\bigr)^{p(n-rs)}. \tag{1.10}$$

Here $\ell(\lambda)$ is the length of the partition λ, z_λ is given by

$$z_\lambda := \prod_{i \in \mathbb{Z}_{\geq 1}} i^{m_i(\lambda)} m_i(\lambda)! \quad \text{ with } \quad m_i(\lambda) := \#\{1 \leq i \leq \ell(\lambda) \mid \lambda_j = i\}, \tag{1.11}$$

and $p(n) := \#\{\lambda \mid \lambda \vdash n\}$ is the partition number of n. To describe the zeros $h_{r,s}$, let us introduce a parametrization of c:

$$c = c(t) := 13 - 6(t + t^{-1}). \tag{1.12}$$

Then the rewritten form $h_{r,s}(t) := h_{r,s}|_{c=c(t)}$ is given by

$$h_{r,s}(t) := \frac{(r-st)^2 - (t-1)^2}{4t}. \tag{1.13}$$

In the following, we often use the pair $(c(t), h)$ for the highest weights. Note that by the symmetry

$$c(t) = c(t^{-1}), \tag{1.14}$$

we have $M(c(t), h) = M(c(t^{-1}), h)$.

The definition (1.3) of the singular vector v indicates that in the expansion $v = \sum_{\lambda \vdash n} c_\lambda L_{-\lambda} |c, h\rangle$ with respect to the basis (1.2), the set of coefficients $(c_\lambda)_{\lambda \vdash n}$ forms an eigenvector of the matrix (1.8) with eigenvalue 0. Thus if a singular vector exists in $M(c, h)$, then the Kac determinant (1.10) vanishes, i.e., the pair (c, h) is expressed as $(c(t), h_{r,s}(t))$ for some $r, s \in \mathbb{Z}_{\geq 1}$ with $rs \leq n$.

1.2. **Explicit formula of singular vectors.** Several studies explored explicit forms of singular vectors. First, let us mention

Fact 1.1 ([18])**.** One can write the singular vector $|\chi_{r,s}\rangle$ on the Verma module $M(c(t), h_{r,s}(t))$ as

$$|\chi_{r,s}\rangle = P_{r,s}(t) \, |c(t), h_{r,s}(t)\rangle ,$$

with

$$P_{r,s}(t) = L_{-1}^{rs} + \cdots \in U(\mathrm{Vir}_-) \otimes \mathbb{C}[t, t^{-1}]. \tag{1.15}$$

The point is that the coefficients in (1.15) are Laurent polynomials of t.

By direct calculations using the matrices (1.9), one can obtain examples for small r and s:

$$\begin{aligned}
P_{1,2}(t) =& P_{2,1}(t^{-1}) = L_{-1}^2 - tL_{-2}, \\
P_{1,3}(t) =& P_{3,1}(t^{-1}) = L_{-1}^3 - 4tL_{-2}L_{-1} + 2t(2t-1)L_{-3}, \\
P_{1,4}(t) =& P_{4,1}(t^{-1}) = L_{-1}^4 - 10tL_{-2}L_{-1}^2 + 9t^2L_{-2}^2 \\
& \qquad + 2t(12t-5)L_{-3}L_{-1} - 6t(6t^2-4t+1)L_{-4}, \\
P_{2,2}(t) =& L_{-1}^4 - 2(t+t^{-1})L_{-2}L_{-1}^2 + (t^2 - 2 + t^{-2})L_{-2}^2 \\
& - 2(t - 3 + t^{-1})L_{-3}L_{-1} - 3(t - 2 + t^{-1})L_{-4}.
\end{aligned} \tag{1.16}$$

In these examples the condition (1.15) is clearly satisfied.

As another simple remark, we have the equality

$$P_{r,s}(t) = P_{s,r}(t^{-1}), \tag{1.17}$$

which is an easy consequence of the symmetries (1.14) and

$$h_{r,s}(t) = h_{s,r}(t^{-1}).$$

In the late 1980s and the early 1990s, a series of works tried to write down $P_{r,s}(t)$ explicitly. [8] gave an explicit formula for $P_{1,s}(t)$. [5] gave an algorithm for constructing general $P_{r,s}(t)$ from $P_{1,s}(t)$. See also [15, §8.A] for this algorithm. [29] gave a formula of $P_{r,s}(t)$ using 'analytic continuation' of $P_{1,s}(t)$. See also [18] for a mathematically rigorous treatment of this 'analytic continuation'. Although these formulas for $P_{r,s}(t)$ were enough for several studies on representation theory (see e.g. [30]), an 'explicit' formula for $P_{r,s}(t)$ could not be obtained.

The paper [34] shed a new light on this problem. It was found that the following two objects coincide up to normalization: the integral expression

of the Jack symmetric function $J^{(t)}_{(s^r)}$ [32, Chap.VI §10] and the expression of $|\chi_{r,s}\rangle$ in terms of the screening operators for the Feigin-Fuchs bosonization of Virasoro algebra [13]. We will recall this topic in §2. See Fact 2.1 in §2.2 for the precise statement.

1.3. **Norm of logarithmic primary.** Let us define an anti-homomorphism

$$\dagger : U(\mathrm{Vir}_-) \to U(\mathrm{Vir}_+), \quad L_{-n} \mapsto L_n.$$

We will also denote this map as $L_{-n}^\dagger = L_n$. Note that $(L_{-\lambda})^\dagger = L_\lambda$ under the notations (1.2) and (1.4). The anti-homomorphism $\dagger$ naturally induces a linear map $M(c,h) \to M^*(c,h)$, which is also written by $\dagger$. Note that $(|c,h\rangle)^\dagger = \langle c,h|$. We define $\langle\chi_{r,s}| := (|\chi_{r,s}\rangle)^\dagger$.

For an element $v \in M(c,h)$, the norm of v is defined to be

$$v^\dagger \cdot v,$$

where $\cdot$ is the contravariant form (1.5). For example, by the definition of singular vector (1.3), it is obvious that

$$\langle\chi_{r,s}|\chi_{r,s}\rangle = \langle c(t), h_{r,s}(t)|\, P^\dagger_{r,s}(t) P_{r,s}(t)\, |c(t), h_{r,s}(t)\rangle = 0.$$

In [25] a curious observation was given on the norm

$$N_{1,s}(t,h) := \langle c(t), h|\, P^\dagger_{1,s}(t) P_{1,s}(t)\, |c(t), h\rangle$$

of the vector $P_{1,s}(t)\,|c(t),h\rangle$. They obtained a formula

$$\begin{aligned} N_{1,s}(t,h) &= (h - h_{1,s}(t)) \cdot R_{1,s}(t) + O((h - h_{1,s}(t))^2), \\ R_{1,s}(t) &:= 2s!(s-1)! \prod_{k=1}^{s-1} (k^2t^2 - 1). \end{aligned} \tag{1.18}$$

A proof (containing some physical discussion) of this factor $R_{1,s}(t)$ was given in [25].

There was a several year gap between the studies of this kind of norm of $P_{r,s}(t)\,|c(t),h\rangle$. One of the reasons why such a calculation did not attract so much interest may be, as a simple matter, that there was no necessity.

In the early 2000s, a revival of the Liouville field theory occurred. Among several important papers, [47, §6] observed a generalization of the formula (1.18). Let us denote

$$N_{r,s}(t,h) := \langle c(t), h|\, P^\dagger_{r,s}(t) P_{r,s}(t)\, |c(t), h\rangle\,. \tag{1.19}$$

Using (1.16), we can calculate some examples:

$$\begin{aligned} N_{1,1}(t,h) =& 2(h - h_{1,1}(t)), \\ N_{1,2}(t,h) =& 4(t^2-1)(h - h_{1,2}(t)) + 8(h - h_{1,2}(t))^2, \\ N_{1,3}(t,h) =& 24(t^2-1)(4t^2-1)(h - h_{1,3}(t)) \\ &+ 8(16t^2-9)(h - h_{1,3}(t))^2 + 48(h - h_{1,3}(t))^3, \\ N_{1,4}(t,h) =& 288(t^2-1)(4t^2-1)(9t^2-1)(h - h_{1,4}(t)) \\ &+ 16(594t^4 - 481t^2 + 66)(h - h_{1,4}(t))^2 \\ &+ 128(25t^2-9)(h - h_{1,4}(t))^3 + 384(h - h_{1,4}(t))^4 \end{aligned}$$

$$\begin{aligned}N_{2,2}(t,h) =& -8(t^2-1)(t^2-4)(t^{-2}-1)(t^{-2}-4)(h-h_{2,2}(t))\\&+16(2t^{-4}-33t^{-2}+91-33t^2+2t^4)(h-h_{2,2}(t))^2\\&+128(t^2-7+t^{-2})(h-h_{2,2}(t))^3+384(h-h_{2,2}(t))^4.\end{aligned}$$

Then it was conjectured that

$$\begin{aligned}&N_{r,s}(t,h) \overset{?}{=} (h-h_{r,s}(t))\cdot R_{r,s}(t)+O((h-h_{r,s}(t))^2),\\&R_{r,s}(t) := 2\prod_{\substack{(k,l)\in\mathbb{Z}^2,\\ 1-r\le k\le r,\ 1-s\le l\le s,\\ (k,l)\neq(0,0),(r,s).}}(kt^{-1/2}+lt^{1/2}).\end{aligned}\tag{1.20}$$

(Note that in [47] $N_{r,s}(t,h)$ is not expanded with respect to h, but α given in (2.4), the Heisenberg counterpart of the highest weight.)

The element $P_{r,s}(t)\,|c(t),h\rangle$ is named the logarithmic primary in [47], so that it is natural to call $N_{r,s}(t)$ the norm of logarithmic primary. This norm is the main object in this paper. The expression (1.20) is a generalization of (1.18). A physical derivation of the factor $R_{r,s}(t)$ was shown in [47] based on the theory of Liouville field on the Poincaré disk [50], but it seems to lack mathematically rigorous arguments. An analogous explanation for the SUSY Liouville field theory was given in [7], but no mathematical proof seems to exist.

As indicated in the last line of [47, §6], the factor $R_{r,s}(t)$ resembles the dominator of the factor appearing in the recursive formula of conformal block given in [45], [46] and [49].

1.4. **AGT relation.** The resemblance of the factor $R_{r,s}(t)$, with the factor appearing in the recursive formula, was recently rediscovered in the context of AGT relations/conjectures.

The original AGT conjecture [2] states an equivalence between the Liouville conformal blocks and the Nekrasov partition functions for $N=2$ supersymmetric SU(2) gauge theories [35]. In [19] degenerated versions of the conjecture were proposed. As the most simplified case, it was conjectured that the inner product of a certain element in the Verma module of Virasoro algebra coincides with the Nekrasov partition function for the four-dimensional pure SU(2) gauge theory.

The element considered is a kind of Whittaker vector in the Verma module of the Virasoro algebra, and now called Gaiotto state. Let us recall its definition. Fix a non-zero complex number Λ. Consider the completed Verma module $\widehat{M}(c,h)$ of $M(c,h)$, where the completion is done with respect to the L_0-weight gradation $M(c,h)=\oplus_{n\in\mathbb{Z}_{\ge 0}}M(c,h)_n$. An element $|G\rangle\in\widehat{M}(c,h)$ is called a Gaiotto state if

$$L_1\,|G\rangle=\Lambda^2\,|G\rangle\,,\quad L_n\,|G\rangle=0\ (n>1).$$

We normalize $|G\rangle$ by the condition

$$|G\rangle=|c,h\rangle+\cdots,$$

which means that the homogeneous component of $|G\rangle$ in $M(c,h)_0$ is $|c,h\rangle$.

On the other hand, the pure SU(r) gauge Nekrasov partition function has the next combinatorial expression (which can be considered as the definition

of the partition function). Let $x, \epsilon_1, \epsilon_2, \overrightarrow{a} = (a_1, a_2, \ldots, a_r)$ be indeterminates.

$$\begin{aligned} Z^{\mathrm{rank}=r}(x;\epsilon_1,\epsilon_2,\overrightarrow{a}) &:= \sum_{\overrightarrow{Y}} \frac{x^{|\overrightarrow{Y}|}}{\prod_{1\le\alpha,\beta\le r} n^{\overrightarrow{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\overrightarrow{a})}, \\ n^{\overrightarrow{Y}}_{\alpha,\beta}(\epsilon_1,\epsilon_2,\overrightarrow{a}) &:= \prod_{\square\in Y_\alpha} [-\ell_{Y_\beta}(\square)\epsilon_1 + (a_{Y_\alpha}(\square)+1)\epsilon_2 + a_\beta - a_\alpha] \\ &\quad \times \prod_{\blacksquare\in Y_\beta} [(\ell_{Y_\alpha}(\blacksquare)+1)\epsilon_1 - a_{Y_\beta}(\blacksquare)\epsilon_2 + a_\beta - a_\alpha]. \end{aligned} \tag{1.21}$$

Here $\overrightarrow{Y} = (Y_1, Y_2, \ldots, Y_r)$ is an r-tuple of partitions, $|\overrightarrow{Y}| := |Y_1| + |Y_2| + \cdots + |Y_r|$, and $a_Y(\square)$, $\ell_Y(\square)$ are the arm and the leg of the box $\square$ with respect to Y as will be defined in (2.7).

Now, the statement of the simplest case of Gaiotto conjectures is

$$\langle G|G\rangle \overset{?}{=} Z^{\mathrm{rank}=2}(x;\epsilon_1,\epsilon_2,\overrightarrow{a}). \tag{1.22}$$

The parameters in both hand sides are related as in Table 1.

Virasoro	Nekrasov
c	$13 + 6(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1)$
h	$\big((\epsilon_1+\epsilon_2)^2 - (a_2-a_1)^2\big)/4\epsilon_1\epsilon_2$
Λ	$x^{1/4}/(\epsilon_1\epsilon_2)^{1/2}$

TABLE 1. Parameter correspondence

Let us also mention the work of [33], which shows

$$\langle G|G\rangle = \sum_{n=0}^{\infty} \Lambda^{4n}(K_n^{-1})_{(1^n),(1^n)},$$

where K_n^{-1} is the inverse matrix of (1.8), and the index '$(1^n), (1^n)$' indicates the position of the element of this inverse matrix (recall that the matrix K_n is indexed by partitions of n). Thus the conjecture (1.22) is equivalent to

$$(K_n^{-1})_{(1^n),(1^n)}(c,h) \overset{?}{=} (\epsilon_1\epsilon_2)^{4n} Z_n(\epsilon_1,\epsilon_2;\overrightarrow{a}). \tag{1.23}$$

1.5. **Proving AGT relation via recursive formula.** There exist several strategies for proving AGT conjectures. As for the pure gauge version (1.23), one of the strategies is to show that both sides satisfy the same recursive formula with respect to n. This strategy was first proposed by [36].

Later, the paper [11] executed the procedure in case of the adjoint matter theory. They used an integral expression of the Nekrasov partition function and showed that it satisfies a recursive formula. The same recursive formula for the Virasoro side was then derived from the Zamolodchikov recursive formula for conformal block by limiting procedure.

Similar arguments were given in [23], where the cases of $N_f = 0, 1, 2$ (the number of matter fields) were treated. Here, we only recall the case

$N_f = 0$, that is, the pure gauge theory case. It was shown that $z_n := (\epsilon_1\epsilon_2)^{4n} Z_n(\epsilon_1, \epsilon_2; \overrightarrow{a})$ satisfies the recursive formula:

$$z_n(t,h) = \delta_{0,n} + \sum_{\substack{(r,s)\in\mathbb{Z}^2_{>0},\\ 1\le rs\le n}} \frac{R_{r,s}(t)^{-1} z_{n-rs}(t, h_{r,s}(t)+rs)}{h - h_{r,s}(t)}. \tag{1.24}$$

Here the formula is written in the Virasoro parameter (t,h). In order to see it in the Nekrasov parameter, one needs to rewrite parameters by Table 1 and $c = c(t)$ defined in (1.12).

On the Virasoro side, it was stated that $f_n(t,h) := (K_n^{-1})_{(1^n),(1^n)}(c(t),h)$ satisfies the following recursive formula:

$$f_n(t,h) = \delta_{0,n} + \sum_{\substack{(r,s)\in\mathbb{Z}^2_{>0},\\ 1\le rs\le n}} \Big[\lim_{h\to h_{r,s}(t)} \frac{N_{r,s}(t,h)}{h-h_{r,s}(t)}\Big]^{-1} \frac{f_{n-rs}(t, h_{r,s}(t)+rs)}{h-h_{r,s}(t)}. \tag{1.25}$$

Actually, one can obtain this formula by considering the Jantzen filtration (see [27], [14] and [10]) on $M(c,h)$ and by some calculation on K_n. Thus from the comparison of (1.24) and (1.25), the verification of the conjecture (1.23) is reduced to the conjecture (1.20).

However, it seems that the justification of (1.20) has not been discussed so far. Note that in the argument of AGT relations, the formula of $R_{r,s}(t)$ (1.20) is cited without proof, only with the remark to the paper [47]. See [11, (1.13a)] and [23, p. 7].

This kind of 'normalization factor' always appears in the recursive formula of conformal blocks and its degenerated versions. See e.g. [48, (5.40)] (Liouville conformal block), [7, (35)] (SUSY Liouville case), [20], [21] and [22] (also SUSY Liouville case).

Therefore, we consider that it is valuable to give a proof of the formula (1.20) for $R_{r,s}$.

1.6. **Main result of this paper.** The main result is a rigorous proof of (1.20). Let us rephrase the statement.

Theorem 1.2. Let

$$P_{r,s}(t) = L_{-1}^{rs} + \cdots \in U(\mathrm{Vir}_-) \otimes \mathbb{C}[t, t^{-1}]$$

be the element which generates the singular vector

$$|\chi_{r,s}\rangle = P_{r,s}(t)\,|c(t), h_{r,s}(t)\rangle$$

in $M(c(t), h_{r,s}(t))$. Define

$$N_{r,s}(t,h) := \langle c(t), h|\, P_{r,s}^{\dagger}(t) P_{r,s}(t)\,|c(t), h\rangle\,.$$

Then $N_{r,s}(t,h)$ has the form

$$N_{r,s}(t,h) = (h - h_{r,s}(t)) \cdot R_{r,s}(t) + O((h-h_{r,s}(t))^2),$$
$$R_{r,s}(t) := 2 \prod_{\substack{(k,l)\in\mathbb{Z}^2,\\ 1-r\le k\le r,\ 1-s\le l\le s,\\ (k,l)\ne(0,0),(r,s).}} (kt^{-1/2} + lt^{1/2}).$$

Let us explain the content of this paper. §2 is devoted to the preliminaries on bosonization and symmetric functions, which are crucial tools in our discussion. In §3 we give the proof of the main Theorem 1.2. Since our argument is rather complicated, the outline is explained in the beginning of this section. We end this paper with §4 giving some remarks on possible generalizations and the related works.

2. Preliminaries on bosonization

2.1. **Bosonization and singular vectors.** Let us recall the Feigin-Fuchs bosonization of Virasoro algebra quickly. Consider the Heisenberg algebra $\mathcal{H}$ generated by a_n ($n \in \mathbb{Z}$) with the relation

$$[a_m, a_n] = m\delta_{m+n,0}.$$

For a fixed $\rho \in \mathbb{C}$, consider the correspondence

$$L_n \mapsto \mathcal{L}_n := \frac{1}{2}\sum_{m\in\mathbb{Z}} {}^{\circ}_{\circ} a_m a_{n-m} {}^{\circ}_{\circ} - (n+1)\rho a_n, \quad C \mapsto 1 - 12\rho^2, \tag{2.1}$$

where the symbol ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ means the normal ordering. This correspondence determines a well-defined morphism

$$\varphi : U(\mathrm{Vir}) \to \widehat{U}(\mathcal{H}).$$

Here $\widehat{U}(\mathcal{H})$ is the completion of the universal enveloping algebra $U(\mathcal{H})$ in the following sense (see also [12], [17] and [16]). For $n \in \mathbb{Z}_{\geq 0}$, let I_n be the left ideal of $U(\mathcal{H})$ generated by all polynomials in a_m ($m \in \mathbb{Z}_{\geq 1}$) of degrees greater than or equal to n (where we defined the degree by $\deg a_m := m$). Then we define

$$\widehat{U}(\mathcal{H}) := \varprojlim_n \widehat{U}(\mathcal{H})/I_n.$$

Next we recall the functorial correspondence of the representations. First let us define the Fock representation $\mathcal{F}_\alpha$ of $\mathcal{H}$. $\mathcal{H}$ has the triangular decomposition of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm := \oplus_{\pm n\in\mathbb{Z}_{\geq 1}} \mathbb{C}a_n$ and $\mathcal{H}_0 := \mathbb{C}a_0$. Let $\mathbb{C}_\alpha = \mathbb{C}\,|\alpha\rangle_{\mathcal{F}}$ be the one-dimensional representation of $\mathcal{H}_0 \oplus \mathcal{H}_+$ with the action $a_0\,|\alpha\rangle_{\mathcal{F}} = \alpha\,|\alpha\rangle_{\mathcal{F}}$ and $a_n\,|\alpha\rangle_{\mathcal{F}} = 0$ ($n \in \mathbb{Z}_{\geq 1}$). Then the Fock space $\mathcal{F}_\alpha$ is defined to be

$$\mathcal{F}_\alpha := \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_0\oplus\mathcal{H}_-} \mathbb{C}_\alpha$$

It has a weight decomposition

$$\mathcal{F}_\alpha = \oplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{F}_{\alpha,n}, \tag{2.2}$$

where each weight space $\mathcal{F}_{\alpha,n}$ has a basis

$$\{a_{-\lambda}\,|\alpha\rangle_{\mathcal{F}} \mid \lambda \vdash n\} \tag{2.3}$$

with $a_{-\lambda} := a_{-\lambda_k} \cdots a_{-\lambda_1}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then the action of $\widehat{U}(\mathcal{H})$ on $\mathcal{F}_\alpha$ is well-defined.

Similarly the dual Fock space $\mathcal{F}^*_\alpha$ is defined to be $\mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_0\oplus\mathcal{H}_-} \mathbb{C}^*_\alpha$, where $\mathbb{C}^*_\alpha = \mathbb{C} \cdot {}_{\mathcal{F}}\langle\alpha|$ is the one-dimensional representation of $\mathcal{H}_0 \oplus \mathcal{H}_-$ with the action ${}_{\mathcal{F}}\langle\alpha|\,a_0 = \alpha \cdot {}_{\mathcal{F}}\langle\alpha|$ and ${}_{\mathcal{F}}\langle\alpha|\,a_{-n} = 0$ ($n \in \mathbb{Z}_{\geq 1}$).

Now we can state the bosonization of representation. (2.1) is compatible with the map

$$\psi : M(c,h) \to \mathcal{F}_\alpha, \quad L_{-\lambda}\,|c,h\rangle \mapsto \mathcal{L}_{-\lambda}\,|\alpha\rangle_{\mathcal{F}}$$

with $\mathcal{L}_{-\lambda} := \mathcal{L}_{-\lambda_1}\mathcal{L}_{-\lambda_2}\cdots\mathcal{L}_{-\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and

$$c = 1 - 12\rho^2, \quad h = \frac{1}{2}\alpha(\alpha - 2\rho). \tag{2.4}$$

In other words, we have

$$\psi(xv) = \varphi(x)\psi(v) \quad (x \in \mathrm{Vir},\ v \in M(c,h))$$

under the parametrization (2.4) of highest weights.

Note that from the parametrization (1.12), (1.13) and the correspondence (2.4), a singular vector occurs at

$$c = 1 - 12\rho(t)^2, \quad h = h_{r,s} = \frac{1}{2}\alpha_{r,s}(t)\big(\alpha_{r,s} - 2\rho(t)\big)$$

with

$$\rho(t) := \frac{1}{\sqrt{2}}(t^{-1/2} - t^{1/2}), \quad \alpha_{r,s}(t) := \frac{1}{\sqrt{2}}\big((r+1)t^{-1/2} - (s+1)t^{1/2}\big). \tag{2.5}$$

The Fock space $\mathcal{F}_\alpha$ is naturally identified with the space of symmetric functions. In this paper, the term symmetric function means the infinite-variable symmetric polynomial. To treat such an object rigorously, we follow the argument of [32, Chap.I §2]. Let us denote by Λ_n the ring of n-variable symmetric polynomials over $\mathbb{Z}$, and by Λ_n^d the space of homogeneous symmetric polynomials of degree d. The ring of symmetric functions Λ is defined as the inverse limit of the Λ_n in the category of graded rings (with respect to the gradation by the degree d). We denote by $\Lambda_K := \Lambda \otimes_{\mathbb{Z}} K$ the coefficient extension to the ring K. Among several bases of Λ, the power sum symmetric function

$$p_n = p_n(x) := \sum_{i \in \mathbb{Z}_{\geq 1}} x_i^n, \quad p_\lambda := p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_k},$$

plays an important role. It is known that $\{p_\lambda \mid \lambda \vdash d\}$ is a basis of $\Lambda_{\mathbb{Q}}^d$, the space of homogeneous symmetric functions of degree d.

Now following [3], we define the isomorphism between $\mathcal{F}_\alpha$ and $\Lambda_{\mathbb{C}(t^{1/2})}$:

$$\iota : \mathcal{F}_\alpha \otimes \mathbb{C}(t^{1/2}) \to \Lambda_{\mathbb{C}(t^{1/2})}, \quad v \mapsto {}_{\mathcal{F}}\langle\alpha|\exp\Big(\frac{1}{\sqrt{2t}}\sum_{n=1}^{\infty}\frac{1}{n}p_n a_n\Big)v.$$

Under this morphism, an element $a_{-\lambda}\,|\alpha\rangle_{\mathcal{F}}$ of the base (2.3) is mapped to

$$\iota(a_{-\lambda}\,|\alpha\rangle_{\mathcal{F}}) = p_\lambda(x)/(\sqrt{2t})^{\ell(\lambda)}.$$

Since $\{p_\lambda\}$ is a basis of $\Lambda_{\mathbb{Q}}$, ι is actually an isomorphism.

Using the examples (1.16) and the Feigin-Fuchs bosonization (2.1), one can calculate some examples of the images of singular vectors:

$$\iota\big(\varphi\big(P_{r,s}(t)\big)\,|\alpha_{r,s}(t)\rangle\big) = \iota\circ\psi\big(P_{r,s}(t)\,|c(t), h_{r,s}(t)\rangle\big) = \iota\circ\psi(|\chi_{r,s}\rangle).$$

The result is

$$\begin{aligned}
\iota \circ \psi(|\chi_{1,1}\rangle) &= J^{(t)}_{(1)} \cdot (t^{-1}-1),\\
\iota \circ \psi(|\chi_{2,1}\rangle) &= J^{(t)}_{(1^2)} \cdot (t^{-1}-1)(2t^{-1}-1),\\
\iota \circ \psi(|\chi_{3,1}\rangle) &= J^{(t)}_{(1^3)} \cdot (t^{-1}-1)(2t^{-1}-1)(3t^{-1}-1),\\
\iota \circ \psi(|\chi_{4,1}\rangle) &= J^{(t)}_{(1^4)} \cdot (t^{-1}-1)(2t^{-1}-1)(3t^{-1}-1)(4t^{-1}-1),\\
\iota \circ \psi(|\chi_{2,2}\rangle) &= J^{(t)}_{(2^2)} \cdot (t^{-1}-1)(t^{-1}-2)(2t^{-1}-1)(2t^{-1}-2).
\end{aligned} \tag{2.6}$$

Here $J^{(t)}_\lambda$ is the integral Jack symmetric function, the definition of which will be recalled in the next subsection. Thus if one expresses $|\chi_{r,s}\rangle$ in terms of the Heisenberg generators a_n's and identifies a_{-n} with the power sum symmetric function p_n, then the expression of $|\chi_{r,s}\rangle$ is proportional to $J^{(t)}_{(s^r)}$.

2.2. **Jack symmetric function.** Now we recall the definition and some properties of Jack symmetric function (see [32, Chap.VI §10] and [40]). Let t be an indeterminate [1] and define an inner product on $\Lambda_{\mathbb{Q}(t)}$ by

$$\langle p_\lambda, p_\mu \rangle_t := \delta_{\lambda,\mu} z_\lambda t^{\ell(\lambda)}.$$

Here z_λ is given in (1.11). Then the monic Jack symmetric function $P^{(t)}_\lambda$ is determined uniquely by the following two conditions:

(i): It has an expansion via monomial symmetric function m_ν in the form

$$P^{(t)}_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}(t) m_\mu.$$

Here $c_{\lambda,\mu}(t) \in \mathbb{Q}(t)$ and the ordering $<$ among the partitions is the dominance semi-ordering.

(ii): The family of Jack symmetric functions is an orthogonal basis with respect to $\langle \cdot, \cdot \rangle_t$:

$$\langle P^{(t)}_\lambda, P^{(t)}_\mu \rangle_t = 0 \quad \text{if } \lambda \neq \mu.$$

In order to define the integral Jack symmetric function $J^{(t)}_\lambda$, it is necessary to express the norm of $P^{(t)}_\lambda$. One can simply write it down using Young diagrams. Following [32] we prepare several notations for diagrams here. To a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, we associate the Young diagram, which is the set of boxes located at $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le k, 1 \le j \le \lambda_i\}$. The coordinate (i,j) is taken so that the index i increases if one reads from top to bottom, and the index j increases if one reads from left to right (see e.g., Figure 1). We will often identify a partition and its associated Young diagram.

Now we define the arm and leg for a box $\square$ located at $(i,j) \in \mathbb{Z}^2_{\ge 1}$ with respect to λ by

$$a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda^\vee_j - i. \tag{2.7}$$

[1] Our parameter t is usually denoted by α in the literature, e.g., in [32]. We avoid using α since it is already defined to be the highest weight of the Heisenberg Fock space $\mathcal{F}_\alpha$.

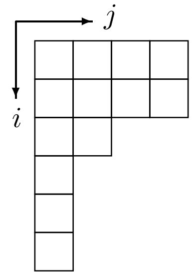

FIGURE 1. The Young diagram for $(4,4,2,1,1,1)$

Here $\lambda^\vee$ is the conjugate partition of λ, which is obtained by transposing the Young diagram of λ. (E.g., for $\lambda = (4,4,2,1,1,1)$ as in Figure 1, we have $\lambda^\vee = (6,3,2,2)$.) We also used the convention $\lambda_i = 0$ for $i > \ell(\lambda)$ and $\lambda_j^\vee = 0$ for $j > \lambda_1$. Thus $a_\lambda(\square)$ and $\ell_\lambda(\square)$ could be minus in general, although such a case does not occur in the norm of Jack symmetric functions. (This generalized arm/leg is necessary for the definition of the Nekrasov partition function (1.21).)

Using these combinatorial notations, one can write down the norm of monic Jack symmetric function as

$$\langle P_\lambda^{(t)}, P_\lambda^{(t)} \rangle_t = \prod_{\square \in \lambda} \frac{t a_\lambda(\square) + \ell_\lambda(\square) + t}{t a_\lambda(\square) + \ell_\lambda(\square) + 1}, \tag{2.8}$$

where $\square \in \lambda$ means that the box $\square$ runs over the boxes in the Young diagram associated to λ.

Then the integral Jack symmetric function $J_\lambda^{(t)}$ is defined to be

$$J_\lambda^{(t)} := P_\lambda^{(t)} \cdot \prod_{\square \in \lambda} (t a_\lambda(\square) + \ell_\lambda(\square) + 1) \tag{2.9}$$

It is known that for a partition λ of n,

$$J_\lambda^{(t)} = \sum_{\mu \vdash n} u_{\lambda,\mu}(t) p_\lambda, \quad u_{\lambda,(1^n)}(t) = 1, \quad u_{\lambda,\mu}(t) \in \mathbb{Z}[t].$$

This is the origin of the word 'integral' Jack symmetric function.

Finally, we can state the fact obtained in [34].

Fact 2.1. (1) [34] The bosonization ψ and the isomorphism ι map the singular vector $|\chi_{r,s}\rangle$ to the integral Jack symmetric function:

$$\iota \circ \psi(|\chi_{r,s}\rangle) \propto J_{(s^r)}^{(t)}.$$

(2) [37] The proportional factor in the above equation is equal to

$$B_{r,s}(t) := \prod_{k=1}^{r} \prod_{l=1}^{s} (k t^{-1} - l). \tag{2.10}$$

3. THE PROOF OF THEOREM 1.2

In this section, we will show our main theorem following the strategy of [25]. Let us set

$$A_{r,s}(t) := \lim_{h \to h_{r,s}(t)} R_{r,s}(t)/(h - h_{r,s}(t)). \tag{3.1}$$

Our proof consists of the following steps.

Step 1: Estimate the degree of $A_{r,s}(t)$ as the Laurent polynomial of t. This step is executed with the help of asymptotic behavior of $P_{r,s}(t)$ studied in [1]. We show these arguments in §3.1.

Step 2: Determine the set S of the zeros of $A_{r,s}(t)$. This step is done in §3.2, and it is divided into three sub-steps:

- Upper bound of $\#S$. The degree estimate in Step 1 gives the upper limit of the number of zeros.
- Lower bound of $\#S$. Next we show that S includes a certain subset S' using bosonization. S' is determined from a coefficient of Jack symmetric functions appearing in the bosonization of singular vector. Then we also show that S is invariant under the action $t \mapsto -t$. Thus S contains $S' \cup -S'$, where $-S' := \{-t \mid t \in S'\}$.
- Third sub-step. Since $\#(S' \cup -S')$ is equal to the upper limit, S should be equal to $S' \cup -S'$.

As the result of Step 1 and Step 2, $A_{r,s}(t)$ is determined up to a numerical factor.

Step 3: Determine the numerical factor of $A_{r,s}(t)$. We can determine this factor from the asymptotic behavior of $P_{r,s}(t)$ with respect to t. This step is done by direct calculation so that we omit the detail.

3.1. **Step 1. Degree estimate.** Let us recall the next fact:

Fact 3.1 ([1])**.** If one expands $P_{r,s}(t)$ as a Laurent polynomial of t, then

$$P_{r,s}(t) = [(r-1)!]^{2s} L_{-r}^{s} t^{-(r-1)s} + \cdots + [(s-1)!]^{2r} L_{-s}^{r} t^{r(s-1)},$$

where '$\cdots$' denotes the intermediate degrees in t.

Now, for a Laurent series $a(t) = \sum_k a_k t^k$ of t, we define the maximum and minimum degrees of $a(t)$ by

$$\max\deg a(t) := \max\{k \mid a_k \neq 0\}, \quad \min\deg a(t) := \min\{k \mid a_k \neq 0\}. \tag{3.2}$$

Lemma 3.2. The maximum and minimum degrees of $A_{r,s}(t)$ are estimated as

$$\max\deg A_{r,s}(t) \leq 2r(s-1), \quad \min\deg A_{r,s}(t) \geq -2(r-1)s.$$

Proof. First we treat $\max\deg A_{r,s}(t)$. Expand $\langle c(t), h| L_s^r L_{-s}^r |c(t), h\rangle$ with respect to h around $h_{r,s}(t)$ as

$$\langle c(t), h| L_s^r L_{-s}^r |c(t), h\rangle = \sum_{k=0}^{rs-1} e_k^{(r,s)}(t)(h - h_{r,s}(t))^k.$$

By Fact 3.1, we have

$$\max\deg A_{r,s}(t) \leq \max\deg \left(e_1^{(r,s)}(t) t^{2r(s-1)}\right).$$

Since $\max\deg e_1^{(r,s)}(t) = 0$ by Lemma 3.13, we have the result.

The estimate for $\min\deg A_{r,s}(t)$ is obtained from that of $\max\deg A_{r,s}(t)$ by using the symmetry $t \mapsto t^{-1}$, $(r,s) \mapsto (s,r)$ (1.17). □

Corollary 3.3. The number of zeros of $A_{r,s}(t)$ is at most $4rs - 2r - 2s$.

3.2. **Step 2. Zero counting.** Recall the bosonization morphism φ (2.1), the correspondence (2.4) and the highest weight (2.5) of Heisenberg algebra at which a singular vector exists. We will use the highest weight shift of Heisenberg algebra:

$$\varepsilon_{r,s}(t,\alpha) := \alpha - \alpha_{r,s}(t), \quad \varepsilon^{\dagger}_{r,s}(t,\alpha) := \alpha - \alpha_{-r,-s}(t).$$

Note that we have

$$h - h_{r,s}(t) = \frac{1}{2}\varepsilon_{r,s}(t,\alpha)\varepsilon^{\dagger}_{r,s}(t,\alpha). \tag{3.3}$$

under the correspondence (2.4).

For a partition λ of n, define the degree of $a_{-\lambda} \in \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-m}]$ by $\deg a_{-\lambda} := |\lambda| = n$, and denote by $\mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-m}]_n$ the subspace of homogeneous elements of degree n. Then $\mathcal{F}_{\alpha,n}$ defined in (2.2) is isomorphic to $\mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-n}]_n \left|\alpha\right\rangle_{\mathcal{F}}$. Similarly, we define the degree of a_λ by $\deg a_\lambda := |\lambda|$ and denote the homogeneous subspace by $\mathbb{C}[a_1, a_2, \dots, a_m]_n$.

Lemma 3.4. (1) We have

$$\varphi(P_{r,s}(c(t),h)) \left|\alpha\right\rangle_{\mathcal{F}} = \varepsilon^{\dagger}_{r,s}(t,\alpha)\Big[g_0(t) + \sum_{k=1}^{rs-1}(\varepsilon_{r,s}(t,\alpha))^k g_k(t)\Big] \left|\alpha\right\rangle_{\mathcal{F}},$$

with

$$\begin{aligned} &g_0(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-rs}]_{rs}, \\ &g_k(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-rs+1}]_{rs} \quad (k \neq 0). \end{aligned}$$

(2) We also have

$${}_{\mathcal{F}}\left\langle\alpha\right| \varphi(P^{\dagger}_{r,s}(c(t),h)) = {}_{\mathcal{F}}\left\langle\alpha\right| \varepsilon_{r,s}(\alpha,t)\Big[g^{\dagger}_0(t) + \sum_{k=1}^{rs-1}(\varepsilon^{\dagger}_{r,s}(t,\alpha))^k g^{\dagger}_k(t)\Big],$$

with

$$\begin{aligned} &g^{\dagger}_0(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_1, a_2, \dots, a_{rs}]_{rs}, \\ &g^{\dagger}_k(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_1, a_2, \dots, a_{rs-1}]_{rs} \quad (k \neq 0). \end{aligned}$$

The point is that a_{-rs} (resp. a_{rs}) only appears in $g_0(t)$ (resp. $g^{\dagger}_0(t)$). Before starting the proof, we show some examples.

Example 3.5.

$$\begin{aligned} &\varphi(P_{1,1}(c(t),h)) \left|\alpha\right\rangle_{\mathcal{F}} = \varepsilon^{\dagger}_{1,1}(t,\alpha) \left|\alpha\right\rangle_{\mathcal{F}}, \\ &\varphi(P_{2,1}(c(t),h)) \left|\alpha\right\rangle_{\mathcal{F}} \\ &\quad = \varepsilon^{\dagger}_{2,1}(t,\alpha)\Big[(1-t)t^{-1}(\sqrt{2}t^{1/2}a_{-1}^2 - a_{-2}) + \varepsilon_{2,1}(t,\alpha)a_{-1}^2\Big] \left|\alpha\right\rangle_{\mathcal{F}}, \\ &\varphi(P_{3,1}(c(t),h)) \left|\alpha\right\rangle_{\mathcal{F}} \\ &\quad = \varepsilon^{\dagger}_{3,1}(t,\alpha)\Big[(1-t)(2-t)t^{-2}(2ta_{-1}^3 - 3\sqrt{2}t^{1/2}a_{-2}a_{-1} + 2a_{-3}) \\ &\qquad\qquad + \varepsilon_{3,1}(t,\alpha)\big((3-2t)\sqrt{2}t^{-1/2}a_{-1}^3 - (4t^{-1}-3)a_{-2}a_{-1}\big) \\ &\qquad\qquad + (\varepsilon_{3,1}(t,\alpha))^2 a_{-1}^3\Big] \left|\alpha\right\rangle_{\mathcal{F}}, \end{aligned}$$

$$\begin{aligned}
&\varphi(P_{4,1}(c(t),h))\,|\alpha\rangle_{\mathcal{F}} \\
&\quad= \varepsilon^{\dagger}_{4,1}(t,\alpha)\Big[(1-t)(2-t)(3-t)t^{-3}\big(2\sqrt{2}t^{3/2}a_{-1}^4+8\sqrt{2}t^{1/2}a_{-3}a_{-1} \\
&\qquad\qquad\qquad\qquad + 3\sqrt{2}t^{1/2}a_{-2}^2-12ta_{-2}a_{-1}^2-6a_{-4}\big) \\
&\qquad + \varepsilon_{4,1}(t,\alpha)\big(2t^{-1}(11-12t+3t^2)a_{-1}^4 \\
&\qquad\qquad - 2\sqrt{2}t^{-3/2}(5-2t)(4-3t)a_{-2}a_{-1}^2 \\
&\qquad\qquad + t^{-2}(9-10t+3t^2)a_{-2}^2 \\
&\qquad\qquad + 2t^{-2}(12-15t+4t^2)a_{-3}a_{-1}\big) \\
&\qquad + (\varepsilon_{4,1}(t,\alpha))^2\big(3\sqrt{2}t^{-1/2}(2-t)a_{-1}^4-2t^{-1}(5-3t)a_{-2}a_{-1}^2\big) \\
&\qquad + (\varepsilon_{4,1}(t,\alpha))^3a_{-1}^4\Big]\,|\alpha\rangle_{\mathcal{F}}\,.
\end{aligned}$$

Proof of Lemma 3.4. (1) $\varphi(P_{r,s}(c(t),h))\,|\alpha\rangle_{\mathcal{F}}$ is a polynomial of degree rs in terms of α. It has at least one zero at $\alpha=\alpha_{-r,-s}$ by the discussion of [13] (see also [42, 31]). Thus the polynomial considered has the form

$$(\alpha-\alpha_{-r,-s}(t))\sum_{k=0}^{rs-1}(\alpha-\alpha_{r,s}(t))^k g_k(t)\,|\alpha\rangle_{\mathcal{F}}\,,$$

where $g_k(t)\in\mathbb{C}[t^{\pm1/2}]\otimes\mathbb{C}[a_{-1},a_{-2},\dots,a_{-rs}]_{rs}$ for any k. Note that the coefficients are in $\mathbb{C}[t^{\pm1/2}]$ by Fact 1.1 and the definition of φ.

Now Lemma 3.14 proved later means that the coefficient of a_{-rs} has degree one as a polynomial of α. Therefore a_{-rs} cannot appear in $g_k(t)$ for $k>1$. Thus we have the consequence.

(2) is similarly shown so that we omit the detail. □

Lemma 3.6. The set of zeros in $A_{r,s}(t)=0$ includes the set of zeros in the coefficient of a_{-rs} in $g_0(t)$.

Proof. Precisely speaking, if the condition holds, then one can show that the state $\varphi(P_{r,s}(c(t),h))\,|\alpha\rangle_{\mathcal{F}}$ vanishes identically. In order to show it, it is enough to show by induction that $a_n\cdot\varphi(P_{r,s}(c(t),h))\,|\alpha\rangle_{\mathcal{F}}=0$ for any n. This is proved in [25, Appendix B]. □

Lemma 3.7. The coefficient $c_\lambda(t)$ of a_{-rs} in $g_0(t)$ is

$$c_\lambda(t)=\prod_{\substack{(k,l)\in\mathbb{Z}^2\setminus\{(r,s)\}\\ 1\le k\le r,\ 1\le l\le s}}(kt^{-1}-l)\cdot\prod_{\substack{(k,l)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ 0\le k\le r-1,\ 0\le l\le s-1}}(lt-k).$$

Proof. Note that by Fact 2.1 and Lemma 3.4 we have

$$(\alpha_{r,s}(t)-\alpha_{-r,-s}(t))\cdot\iota(g_0(t)\,|\alpha\rangle_{\mathcal{F}})=B_{r,s}(t)J^{(t)}_{(s^r)}.$$

On the other hand, by Corollary 3.16 in §3.3 we have

$$J^{(t)}_{(s^r)}=\sum_{\mu\vdash rs}\theta^{\mu}_{(s^r)}(t)p_\mu,\quad \theta^{(rs)}_{(s^r)}(t)=\prod_{\substack{(k,l)\in\mathbb{Z}^2\setminus\{(0,0)\}\\ 0\le k\le r-1,\ 0\le l\le s-1}}(lt-k).$$

Since $\iota(a_{-rs}\,|\alpha\rangle_{\mathcal{F}}) = p_{rs}/\sqrt{2t}$, we have

$$(\alpha_{r,s}(t) - \alpha_{-r,-s}(t))c_\lambda(t)/\sqrt{2t} = B_{r,s}(t)\cdot\theta_\lambda^{(n)}(t).$$

Then an easy calculation shows the statement. $\square$

Corollary 3.8. $A_{r,s}(t,h)$ has zeros at

$$\begin{aligned} S' &:= \{t = k/l \mid 1 \le k \le r,\ 1 \le l \le s,\ (k,l) \ne (r,s)\} \\ &\quad \cup \{t = k/l \mid 1 \le k \le r-1,\ 1 \le l \le s-1\} \quad \text{(multiplicities included).} \end{aligned}$$

In particular, $A_{r,s}(t,h)$ has at least $\# S' = 2rs - r - s$ zeros.

Proof. This is the consequence of Lemma 3.6 and Lemma 3.7. $\square$

Lemma 3.9. $A_{r,s}(t) = A_{r,s}(-t)$.

Proof. In fact one can show a stronger statement: if one expands

$$N_{r,s}(t,h) = \sum_{k=1}^{rs} (h - h_{r,s}(t))^k n_{r,s,k}(t), \tag{3.4}$$

then we have $n_{r,s,k}(t) = n_{r,s,k}(-t)$. $A_{r,s}(t)$ is nothing but $n_{r,s,1}(t)$. This is the consequence of the following Fact 3.10 stated in [14, Theorem 1.13] (for a proof, see also [30]). $\square$

Fact 3.10 ([14, Theorem 1.13])**.** The anti-automorphism on $U(\mathrm{Vir}_-)[t,t^{-1}]$ defined by

$$t \mapsto -t, \quad L_{-i} \mapsto (-1)^{i-1} L_{-i}$$

takes $P_{r,s}(t)$ into itself.

Corollary 3.11. Let us denote $-S' := \{-t \mid t \in S'\}$. Then $A_{r,s}(t)$ has zeros at $S' \cup (-S')$ (multiplicities included). Thus $A_{r,s}(t)$ has at least $2 \cdot \# S = 4rs - 2r - 2s$ zeros.

Proof. This is the consequence of Corollary 3.8 and Lemma 3.9. $\square$

Corollary 3.12. $A_{r,s}(t)$ is equal to

$$A'_{r,s}(t) := \prod_{\substack{(k,l)\in\mathbb{Z}^2,\\ 1-r\le k\le r,\ 1-s\le l\le s,\\ (k,l)\ne(0,0),(r,s).}} (kt^{-1/2} + lt^{1/2})$$

up to a numerical factor in $\mathbb{C}$.

Proof. By Corollary 3.3 and Corollary 3.11, we have known that the set S of zeros of $A_{r,s}(t)$ is equal to $S' \cup (-S')$. Then the inequalities in Lemma 3.2 should be equalities: $\max\deg A_{r,s}(t) = 2r(s-1)$, $\min\deg A_{r,s}(t) = -2(r-1)s$.

Next note that $A'_{r,s}(t) \in \mathbb{C}[t,t^{-1}]$, that the set of its zeros is equal to $S' \cup (-S')$, and that $\max\deg A'_{r,s}(t) = 2r(s-1)$, $\min\deg A'_{r,s}(t) = -2(r-1)s$. Thus $A_{r,s}(t)$ and $A'_{r,s}(t)$ should be proportional. $\square$

3.3. **Lemmas for the proof.**

3.3.1. *A lemma on Shapovalov form.*

Lemma 3.13. (1)

$$\langle c,h|\, L_s^r L_{-s}^r \,|c,h\rangle = s^r r! \prod_{k=1}^{r} [2h + s(k-1) + \frac{s^2-1}{12}c]$$

(2) In the expansion

$$\langle c(t),h|\, L_s^r L_{-s}^r \,|c(t),h\rangle = \sum_{k=0}^{s} (h - h_{r,s}(t))^k e_k^{(r,s)}(t),$$

the maximum and the minimum degrees (see (3.2) for the definitions) of $e_1^{(r,s)}(t)$ are

$$\max\deg e_1^{(r,s)}(t) = 0, \quad \min\deg e_1^{(r,s)}(t) = \begin{cases} 0 & r = s \\ 1-r & r \neq s \end{cases}.$$

Proof. By the defining relations of Vir we have

$$\langle c,h|\, L_s^r L_{-s}^r \,|c,h\rangle = \langle c,h|\, L_s^{r-1} L_{-s}^{r-1} \,|c,h\rangle \cdot \sum_{k=1}^{r} \Big[\frac{s^3-s}{12}c + 2s(sk+h)\Big].$$

Then an easy calculation gives the result of (1).

(2) is an immediate consequence of (1). □

3.3.2. *A lemma on bosonization.*

Lemma 3.14. Let λ be a partition of n. In the expansion

$$\varphi(L_{-\lambda})\,|\alpha\rangle_{\mathcal{F}} = \sum_{\mu\vdash n} c_\lambda^\mu(t,\alpha) a_{-\mu}\,|\alpha\rangle_{\mathcal{F}} \quad c_\lambda^\mu(t,\alpha) \in \mathbb{C}[\alpha, t^{\pm 1/2}],$$

the coefficient $c_\lambda^{(n)}(t,\alpha)$ of $a_{-(n)}\,|\alpha\rangle_{\mathcal{F}}$ is of degree one in terms of α.

Proof. This is an easy consequence of the bosonization φ:

$$\begin{aligned}\varphi(L_{-2n}) &= (2n-1)\rho(t)a_{-2n} + \frac{a_{-n}^2}{2} + a_{-n-1}a_{-n+1} + a_{-n-2}a_{-n+2} + \\ &\qquad \cdots + a_{-2n+1}a_{-1} + a_{-2n}a_0 + a_{-2n-1}a_1 + \cdots, \\ \varphi(L_{-2n-1}) &= 2n\rho(t)a_{-2n} + a_{-n-1}a_{-n} + a_{-n-2}a_{-n+1} + \\ &\qquad \cdots + a_{-2n}a_{-1} + a_{-2n-1}a_0 + a_{-2n-2}a_1 + \cdots.\end{aligned}$$

In fact, for the case $\ell(\lambda) = 1$, i.e., $\lambda = (n)$, the above expression of $\varphi(L_{-n})$ gives $\varphi(L_{-n})\,|\alpha\rangle_{\mathcal{F}} = (\alpha + (n-1)\rho(t))a_n\,|\alpha\rangle_{\mathcal{F}} + \cdots$, which is the desired consequence.

For the case $k := \ell(\lambda) > 1$, set $\nu := (\lambda_2, \lambda_3, \ldots, \lambda_k)$. Since

$$\varphi(L_{-\nu})\,|\alpha\rangle_{\mathcal{F}} = \sum_{\mu\vdash n-\lambda_1} c_\nu^\mu(t,\alpha) a_{-\mu}\,|\alpha\rangle_{\mathcal{F}},$$

we find that the term $a_{-n}\,|\alpha\rangle_{\mathcal{F}}$ in $\varphi(L_{-\lambda})\,|\alpha\rangle_{\mathcal{F}} = \varphi(L_{-\lambda_1})\varphi(L_{-\nu})\,|\alpha\rangle_{\mathcal{F}}$ appears as

$$a_{-n}a_{n-\lambda_1} \cdot c_\nu^{(n-\lambda_1)}(t,\alpha) a_{-n+\lambda_1}\,|\alpha\rangle_{\mathcal{F}} = (n-\lambda_1) c_\nu^{(n-\lambda_1)}(t,\alpha) a_{-n}\,|\alpha\rangle_{\mathcal{F}}.$$

By the induction hypothesis, we know that $c_\nu^{(n-\lambda_1)}(t,\alpha)$ is of degree one as a polynomial of α. Thus the desired consequence holds. □

3.3.3. *A coefficient of Jack symmetric function.*

Fact 3.15. If one expands the power symmetric function p_n by the family of monic Jack symmetric functions $\{P_\lambda^{(t)}\}$, then

$$p_n = nt \sum_{\lambda \vdash n} \prod_{\square \in \lambda} \frac{1}{t a_\lambda(\square) + \ell_\lambda(\square) + t} \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} (t(j-1) - (i-1)) \cdot P_\lambda^{(t)}.$$

(For the notations of Young diagrams, see §2.2.)

Proof. See [24]. □

Corollary 3.16. Let λ be a partition of n. If one expands the integral Jack symmetric function $J_\lambda^{(t)}$ by the family of power symmetric function $\{p_\mu\}$ as

$$J_\lambda^{(t)} = \sum_{\mu \vdash n} \theta_\lambda^\mu(t) p_\mu$$

then the following holds:

$$\theta_\lambda^{(n)}(t) = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(1,1)}} ((j-1)t - (i-1)).$$

Proof. The coefficient c_λ can be calculated by Fact 3.15, the ratio (2.9) of $P_\lambda^{(t)}$ and $J_\lambda^{(t)}$, and the norm (2.8) of $P_\lambda^{(t)}$. □

4. Conclusion and Remarks

The main result of this paper is Theorem 1.2, a mathematical proof of (1.20). In our proof, important roles are played by Feigin-Fuchs bosonization and Jack symmetric functions. The technical but crucial point of our discussion is Fact 2.1 that the bosonized singular vector of Virasoro algebra is proportional to Jack symmetric function. There are analogous phenomena of this fact in other algebras, such as $\mathcal{W}$ algebras and the deformed Virasoro algebra. So it may be possible to simulate Theorem 1.2 for these algebras. Let us discuss the possibility of such extensions and also mention to the related AGT conjectures.

The paper [3] showed that singular vectors of $\mathcal{W}(\mathfrak{sl}_n)$ algebra are proportional to Jack symmetric functions associated to general (i.e., not necessarily rectangle) partitions. But the proportional factor is not known so far, which is an obstruction to simulate our strategy to calculate the norm of logarithmic primaries in $\mathcal{W}$ algebra case.

Let us also mention to the SU(n) AGT conjecture, where $\mathcal{W}(\mathfrak{sl}_n)$ algebra appears (see e.g. [43]). In [41] a pure gauge AGT conjecture for $\mathcal{W}(\mathfrak{sl}_3)$ algebra was proposed. In this case, one can observe Zamolodchikov-type recursive formula, so that the strategy for the proof of the conjecture may be built. However in order to execute this strategy, it is necessary to overcome the obstruction mentioned above . Note that the 'finite analog' of this AGT conjecture is solved recently by methods in geometric representation theory [9].

Another possibility is the proof of the q-deformed/five-dimensional AGT conjecture proposed in [4]. It is conjectured that the K-theoretic Nekrasov

partition function coincides with the norm of the Gaiotto state in the Verma module of the deformed Virasoro algebra. In [44] the author gave a recursive formula for the K-theoretic Nekrasov partition function. However, our knowledge on the singular vectors is not enough to give some proof of the recursive formula in the deformed Virasoro side. It is known that the singular vectors of deformed Virasoro algebra are proportional to Macdonald symmetric functions associated to rectangle partitions [39], but their proportional factors are not known. Also the degree estimation of the norm of logarithmic primaries in the deformed case is not known.

Acknowledgements. The author is supported by JSPS Fellowships for Young Scientists (No.21-2241) and JSPS/RFBR joint project 'Integrable system, random matrix, algebraic geometry and geometric invariant'. Results in this paper were presented at the workshops 'Topics on BC systems and AGT conjectures', Tokyo, September 2010, and 'Synthesis of integrabilities in the context of duality between strings and gauge theories', Moscow, September 2010. Thanks are due to the organizers and participants of these conferences for stimulating discussion. The author also expresses gratitude to the adviser Professor Kōta Yoshioka and Professor Yasuhiko Yamada for the valuable discussion.

References

[1] Astashkevich, A., Fuchs, D.: Asymptotics for singular vectors in Verma modules over the Virasoro algebra. Pacific J. Math. **177**, no. 2, 201–209 (1997).

[2] Alday L. F., Gaiotto D., Tachikawa Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. **91**, 167–197 (2010).

[3] Awata, H., Matsuo, Y., Odake, S., Shiraishi, J.: Excited states of the Calogero-Sutherland model and singular vectors of the W_N algebra. Nucl. Phys. B **449**, 347–374 (1995).

[4] Awata, H., Yamada, Y.: Five-dimensional AGT conjecture and the deformed Virasoro algebra. JHEP 1001:125 (2010).

[5] Bauer, M., Di Francesco, Ph., Itzykson, C., Zuber, J. B.: Singular vectors of the Virasoro algebra. Phys. Lett. **B 260**, no. 3-4, 323–326 (1991).

[6] Belavin, A. A., Polyakov, A. M., Zamolodchikov, A. B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. **B 241**, no. 2, 333–380 (1984).

[7] Belavin A., Zamolodchikov Al. B.: Higher equations of motion in N = 1 SUSY Liouville field theory. JETP Lett. **84**, 418–424 (2006). arXiv: hep-th/0610316.

[8] Benoit, L., Saint-Aubin, Y.: Degenerate conformal field theories and explicit expressions for some null vectors. Phys. Lett. **B 215**, no. 3, 517–522 (1988).

[9] Braverman, A., Feigin, B., Rybnikov, L., Finkelberg, M: A finite analog of the AGT relation I: finite W-algebras and quasimaps' spaces. arXiv:1008.3655.

[10] Etingof, P., Styrkas, K.: Algebraic integrability of Schrödinger operators and representations of Lie algebras. Compositio Math. **98**, no. 1, 91–112 (1995).

[11] Fateev, V. A., Litvinov, A. V.: On AGT conjecture. JHEP 1002:014 (2010).

[12] Feigin, B. L., Frenkel, E.: Quantum $\mathcal{W}$-algebras and elliptic algebras. Commun. Math. Phys. **178**, 653–678 (1996) .

[13] Feigin, B. L., Fuchs, D. B.: Skew-symmetric invariant differential operators on the line and Verma modules over the Virasoro algebra. Funct. Anal. Appl. **16**, 47–63, 96 (1982).

[14] Feigin, B. L., Fuchs, D. B.: Representations of the Virasoro algebra. In: Representation of Lie groups and related topics, 465–554, Adv. Stud. Contemp. Math. **7**, Gordon and Breach (1990).

[15] Di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal field theory. Graduate Texts in Contemporary Physics. Springer-Verlag (1997).

[16] Frenkel, E.: Langlands correspondence for loop groups. Cambridge Studies in Advanced Mathematics, **103**. Cambridge University Press (2007).

[17] Frenkel, E., Ben-Zvi, D.: Vertex algebras and algebraic curves, 2nd edn. Mathematical Surveys and Monographs, **88**. Amer. Math. Soc. (2004).

[18] Fuchs, D. B.: Singular vectors over the Virasoro algebra and extended Verma modules. In: Unconventional Lie algebras, 65–74, Adv. Soviet Math., **17**, Amer. Math. Soc. (1993).

[19] Gaiotto, D.: Asymptotically free $N = 2$ theories and irregular conformal blocks. arXiv:0908.0307 [hep-th].

[20] Hadasz, L., Jaskólski, Z., Suchanek, P.: Recursion representation of the Neveu-Schwarz superconformal block. JHEP 0703:032 (2007).

[21] Hadasz, L., Jaskólski, Z., Suchanek, P.: Elliptic recurrence representation of the $N = 1$ Neveu-Schwarz blocks. Nucl. Phys. **B 798**, 363-378 (2008).

[22] Hadasz, L., Jaskólski, Z., Suchanek, P.: Elliptic recurrence representation of the $N = 1$ superconformal blocks in the Ramond sector. JHEP 0811:060 (2008).

[23] Hadasz, L., Jaskólski, Z., Suchanek, P.: Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals. JHEP 1006:046 (2010).

[24] Hanlon, P. J., Stanley, R. P., Stembridge, J. R.: Some combinatorial aspects of the spectra of normally distributed random matrices. In: Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991), 151–174, Contemp. Math., **138**, Amer. Math. Soc. (1992).

[25] Imbimbo, C., Mahapatra, S., Mukhi S.: Construction of physical states of nontrivial ghost number in $c < 1$ string theory. Nuclear Phys. **B 375**, no. 2, 399–420 (1992).

[26] Itzykson, C., Drouffe, J. M.: Statistical field theory. Vol. 2. Strong coupling, Monte Carlo methods, conformal field theory, and random systems. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge (1989).

[27] Jantzen, J. C.: Moduln mit einem höchsten Gewicht. Lect. Notes in Math. **750**, Springer (1979).

[28] Kac, V. G.: Contravariant form for infinite-dimensional Lie algebras and superalgebras. Lect. Notes in Phys. **94**, 441–445 (1979).

[29] Kent, A.: Singular vectors of the Virasoro algebra. Phys. Lett. **B 273**, no. 1-2, 56–62 (1991).

[30] Kent, A.: Projections of Virasoro singular vectors. Phys. Lett. **B 278**, no. 4, 443–448 (1992).

[31] Kato, M., Matsuda, S.: Null field construction in conformal and superconformal algebras. In: Conformal field theory and solvable lattice models (Kyoto, 1986), 205–254, Adv. Stud. Pure Math., **16**, Academic Press, San Diego (1988).

[32] Macdonald, I. G.: Symmetric functions and Hall polynomials. 2nd ed. Oxford Mathematical Monographs, Oxford University Press (1995).

[33] Marshakov, A., Mironov, A., Morozov, A.: On non-conformal limit of the AGT relations. Phys. Lett. B **682**, 125–129 (2009).

[34] Mimachi, K., Yamada, Y.: Singular vectors of the Virasoro algebra in terms of Jack symmetric polynomials. Commun. Math. Phys. **174**, 447–455 (1995).

[35] Nekrasov, N. A.: Seiberg-Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**, no. 5, 831–864 (2003).

[36] Poghossian, R.: Recursion relations in CFT and $N = 2$ SYM theory JHEP 0912:038 (2009).

[37] Sakamoto, R., Shiraishi, J., Arnaudon, D., Frappat, L., Ragoucy, E.: Correspondence between conformal field theory and Calogero-Sutherland model. Nucl. Phys. **B 704**, 490-509 (2005).

[38] Shiraishi, J.: Lectures on Quantum Integrable Systems (in Japanese). SGC Library vol **28**, Saiensu-sha, (2003).

[39] Shiraishi, J., Kubo, H., Awata, H., Odake, S.: A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions. Lett. Math. Phys. **38**, no. 1, 33–51 (1996).

[40] Stanley, R. P.: Some combinatorial properties of Jack symmetric functions. Adv. Math. **77**, no. 1, 76–115 (1989).

[41] Taki., M.: On AGT Conjecture for Pure Super Yang-Mills and W-algebra. arXiv: 0912.4789 [hep-th].
[42] Tsuchiya, A., Kanie, Y.: Fock space representations of the Virasoro algebra. Intertwining operators. Publ. Res. Inst. Math. Sci. **22**, no. 2, 259–327 (1986).
[43] Wyllard, N.: A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2$ SU(N) quiver gauge theories. JHEP 0911:002 (2009).
[44] Yanagida, S.: Five-dimensional SU(2) AGT conjecture and recursive formula of deformed Gaiotto state. J. Math. Phys. **51**, 123506 (2010).
[45] Zamolodchikov, Al. B.: Conformal symmetry in two dimensions: an explicit recurrence formula for the conformal partial wave amplitude. Comm. Math. Phys. **96**, no. 3, 419–422 (1984).
[46] Zamolodchikov, Al. B.: Conformal symmetry in two-dimensional space: on a recurrent representation of the conformal block. Teoret. Mat. Fiz. **73**, no. 1, 103–110 (1987).
[47] Zamolodchikov, Al. B.: Higher equations of motion in Liouville field theory. Proceedings of 6th International Workshop on Conformal Field Theory and Integrable Models. Internat. J. Modern Phys. **A 19**, May, suppl., 510–523 (2004). arXiv: hep-th/0312279.
[48] Zamolodchikov, A. B., Zamolodchikov, Al. B.: Conformal field theory and 2-D critical phenomena. 3. Conformal bootstrap and degenerate representations of conformal algebra. ITEP-90-31, preprint (1990).
[49] Zamolodchikov, A. B., Zamolodchikov, Al. B.: Conformal bootstrap in Liouville field theory. Nuclear Phys. **B 477**, no. 2, 577–605 (1996).
[50] Zamolodchikov, A. B., Zamolodchikov, Al. B.: Liouville field theory on a pseudosphere. arXiv: hep-th/0101152.

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